Lesson Plan

Students will be able to:

- Analyze functions for intervals of continuity or points of discontinuity.
- Determine the applicability of important calculus theorems using continuity.

Assessment:

- In Class Monitoring
- Learnerator
- MC Exit Slip

Activities:

- Investigation (Evaluating)
- Video Clip (Understanding)
- Guided Notes (Understanding)
- Interactive Digital Card Sort (Analyzing)
EU 1.2 Continuity is a key property of functions that is defined using limits.
Continuity

Most of the techniques of calculus require that functions be continuous.

A function is continuous if you can draw it in one motion without picking up your pencil.

In Calculus, we typically look at continuity at particular points.
Discontinuous at

\[ x = \]

\[ x = \]

Continuous at

\[ x = \]

\[ x = \]

\[ x = \]
Continuity at an Interior Point

In order for a function $f(x)$ to be continuous at a point $x = c$, it must fulfill **ALL THREE** of the following conditions.

- $f(c)$ exists (defined)
- $\lim_{x \to c} f(x)$ exists (left hand = right hand)
- $\lim_{x \to c} f(x) = f(c)$

Limit exists + function defined at the limit = continuous
Continuity at an Endpoint

A function is continuous at left endpoint, a, or right endpoint, b, of its domain if

\[
\lim_{{x \to a^+}} f(x) = f(a) \quad \text{or} \quad \lim_{{x \to b^-}} f(x) = f(b)
\]

The endpoint has a closed circle
Using definition of continuity, determine if the \( f(x) \) is continuous at the following points:

- \( x = -3 \)
- \( x = -1 \)
- \( x = 0 \)
- \( x = 1 \)
Is \( f(x) \) continuous at \( x = 2 \)?

1. \( f(x) = \begin{cases} 
  x + 1 & x < 2 \\
  2x - 1 & x \geq 2 
\end{cases} \)

2. \( f(x) = \begin{cases} 
  x + 1 & x < 2 \\
  2x - 1 & x > 2 
\end{cases} \)

3. \( f(x) = \begin{cases} 
  x + 1 & x < 2 \\
  2x + 1 & x \geq 2 
\end{cases} \)

4. \( f(x) = \begin{cases} 
  x + 1 & x < 2 \\
  x^2 & x = 2 \\
  2x - 1 & x > 2 
\end{cases} \)
Types of Discontinuities

- Removable Discontinuity
- Jump
- Infinite
- Oscillating
Jump Discontinuity

A **jump discontinuity** is typically caused by a piecewise-defined function whose pieces don’t meet neatly.

\[
\begin{cases}
  x + 1, & x \geq 0 \\
  -x + 3, & x < 0
\end{cases}
\]

\[
\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)
\]
Removable Discontinuities

A **removable discontinuity** (hole) occurs when you have a rational expression with common factors in the numerator and denominator. Because these factors can be canceled, the discontinuity is “removable.”

\[
\begin{align*}
\text{f(x)} &= \frac{x^2 - x - 12}{x - 4} \\
\text{f(x)} &= \frac{(x + 3)(x - 4)}{x - 4} \\
\text{f(x)} &= x + 3
\end{align*}
\]
Removable Discontinuity

A **removable discontinuity** also occurs when the curve has a “hole” in it from a missing point because the function has a value at that point that’s “off the curve.”

\[ \lim_{{x \to a}} f(x) \neq f(a) \]

\[ f(x) = \begin{cases} 
  x + 1 & x \neq 2 \\
  -1 & x = 2 
\end{cases} \]
An infinite discontinuity occurs when the curve has a vertical asymptote.

\[ f(x) = \frac{3}{x + 4} \]
Find any discontinuities and determine the type

1. \( f(x) = \begin{cases} 
  x + 3 & x \leq 2 \\
  x^2 & x > 2 
\end{cases} \)

2. \( f(x) = \begin{cases} 
  \frac{x^2 - 4}{x - 2} & x \neq 2 \\
  4 & x = 2 
\end{cases} \)

3. \( f(x) = \frac{5}{x - 2} \)

4. \( f(x) = \frac{x^2 - 8x + 15}{x^2 - 6x + 5} \)

5. \( f(x) = \begin{cases} 
  1 & x \leq -1 \\
  -x & -1 < x < 0 \\
  1 & x = 0 \\
  -x & 0 < x < 1 \\
  1 & x \geq 1 
\end{cases} \)

1. jump disc @ \( x = 2 \)
2. cont @ \( x = 2 \)
3. inf disc @ \( x = 2 \)
4. rem @ \( x = 5 \), inf @ \( x = 1 \)
5. removable @ \( x = 0 \), jump disc @ \( x = 1 \)
Find the value of $c$ that makes the function continuous.

1. $f(x) = \begin{cases} 
  cx + 5 & x < 4 \\
  x^2 - x & x \geq 4 
\end{cases}$

2. $f(x) = \begin{cases} 
  x^2 - c & x < 5 \\
  4x + 2c & x \geq 5 
\end{cases}$

3. $f(x) = \begin{cases} 
  x^3 & x \leq c \\
  x^2 & x > c 
\end{cases}$

1. $c = \frac{4}{7}$
2. $c = \frac{5}{3}$
3. $c = 0, 1$
The Intermediate Value Theorem

by Desmos | 15-30 minutes | Development

This activity asks students to speculate on the existence of roots based on graphs of functions. This is intended as a first exposure to the ideas of the Intermediate Value Theorem, and to provide fodder for classroom conversation about important ideas behind the theorem, including:

(1) The "continuous" condition is essential (though so very often overlooked by students), and
(2) The Intermediate Value Theorem is silent on where there are NOT roots, only on where there ARE roots (given the right conditions).
Intermediate Value Theorem

If a function $f(x)$ is continuous on the closed interval $[a,b]$ then for every real number $d$ between $f(a)$ and $f(b)$, there exists a $c$ between $a$ and $b$ such that $f(c) = d$. 
\textbf{Date vs Weight Graph}

\begin{itemize}
  \item $a = \text{Dec } 1^{st}$ \& $b = \text{Dec } 30^{th}$
  \item $f(a) = 180$ \& $f(b) = 191$
  \item pick a $y$ value between 180 and 191
  \item that $y$ value has to correspond to a date between Dec $1^{st}$ and Dec $30^{th}$.
\end{itemize}
Verify the IVT, find \( c \).

1. \( f(x) = x^2 + x - 1 \) on \([0,5]\) and \( f(c) = 11\)

\[
\begin{align*}
  a &= 0 & f(0) &= -1 \\
  b &= 5 & f(5) &= 29 \\
\end{align*}
\]

Because \( f(0) < f(c) < f(5) \),
\( c \) has to be in the interval \( 0 < c < 5 \)
by the IVT

\[
\begin{align*}
  11 &= x^2 + x - 1 \\
  0 &= x^2 + x - 12 \\
  0 &= (x + 4)(x - 3) \\
  x &= -4, 3
\end{align*}
\]
Verify the IVT, find $c$.

2. $f(x) = x^2 - x - 4$ on $[0,6]$ and $f(c) = 2$

$a = 0 \quad f(0) = -4$

$b = 6 \quad f(6) = 26$

Because $f(0) < f(c) < f(6)$,

c has to be in the interval $0 < c < 6$

by the IVT

$2 = x^2 - x - 4$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

$x = 3, -2$
FR – Limits

FR1: The position function $s(t) = -4.9t^2 + 396.9$ gives the height (m) of an object that has fallen from a height of 396.9 meters after $t$ seconds.

a) Explain why there must exist a time $t$, $1 < t < 2$, at which the height of the object must be 382 meters above the ground.

Because $s(t)$ is a continuous function and $s(1) = 392$ and $s(2) = 377.3$ there exists a time $t$, $1 < t < 2$, where $s(t)$ is a value between 377.3 and 392 using the IVT.
MC – Limits

MC4: If a function is defined by \( f(x) \) below and is continuous for all values of \( x \) on the interval \([0,6]\). Which of the following is the value of \( b \)?

\[
f(x) = \begin{cases} 
1 + e^{-2x} & 0 < x \leq b \\
1 + e^{2x-12} & b < x \leq 6 
\end{cases}
\]

a. 1  b. 2  c. 3  d. 4
MC – Limits

MC5: If $f$ is a continuous function defined by

$$f(x) = \begin{cases} 
  x^2 - bx & x \leq 3 \\
  3 \cos\left(\frac{\pi}{4} x\right) & x > 3 
\end{cases}$$

a. -2.293  
   b. 2.293  
   c. -3.707  
   d. 3.707
MC – Limits

MC6: Let $f$ be a continuous function on the closed interval $[-3, 6]$. If $f(-3) = -1$ and $f(6) = 3$, then the Intermediate Value Theorem guarantees that

a. $f(0) = 0$

b. $-1 \leq f(x) \leq 3$ for all $x$ between $-3$ & $6$

c. $f(c) = 1$ for at least one $c$ between $-3$ & $6$

da. $f(c) = 0$ for at least one $c$ between $-1$ & $3$
Calculus AB: Continuity. Students create functions with certain stipulations.